Spectral Characterization of Dielectric Constant Fluctuation in Hypersonic Wake Plasmas

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Formal procedure and critical steps leading to the determination of a spectrum function in the Booker scattering formula (first-order Born approximation) for dielectric constant fluctuations in the turbulent wake of hypersonic objects are examined. It is concluded that even though such a spectrum function can always be unambiguously defined, our present knowledge about the statistical properties of chemically reacting turbulent wakes generated by hypersonic objects is still much too inadequate to allow accurate determination of such a function. In particular, it is found that in the presence of rapid electron attachment and negative ion formation, the spectrum function depends not only on autocorrelation of local electron density fluctuations, as commonly assumed, but also on cross correlations among electron-, ion-, and mass-density fluctuations, and on autocorrelation of the latter two fluctuations. The relative magnitudes of these additional correlation terms are estimated. Probable effects of these and other complications on the general problem of spectrum function determination for hypersonic wake plasmas are also discussed.

Nomenclature

A = normalization constant
a_0 = radius of the first Bohr orbit
c = velocity of light
$\bar{C}_e$ = mean thermal speed of free electrons
e = charge on electron
E = turbulence energy spectrum
$\mathbf{E}_0$ = electric field vector of incident electromagnetic wave
F = spectrum function pertaining to scattering
g = form factor as defined by Eq. (53)
h = Planck's constant
I = ratio between mass-density fluctuation intensity an electron-density fluctuation intensity as defined by Eq. (38)
k = Boltzmann constant
$\mathbf{k}_0, \mathbf{k}$ = propagation vector for incident and scattered electromagnetic wave
K = complex electrical conductivity
l_{De} = Debye length
m = mass per particle
n = number density of particle (with subscript)
n = refractive index (without subscript)
N = number of members in statistical ensemble; also for complex index of refraction
p = pressure
q = wave number in spectrum function
q_K = Kolmogorov wave number
q^* = wave number for spectral maximum
$\mathbf{q}$ = $\mathbf{k} - \mathbf{k}_0$ = difference between scattered and incident wave vectors
$\bar{Q}_\zeta$ = averaged momentum transfer cross section between electron and type ζ molecule
$\mathbf{r}_1, \mathbf{r}_2$ = position vectors at arbitrary points 1 and 2
$\mathbf{r}'$ = $\mathbf{r}_2 - \mathbf{r}_1$ = relative position vector between two points
r_n = radius of hypersonic sphere
Ry = Rydberg frequency
Re_∞ = flight Reynolds number of hypersonic object
ΔR = length of radar range gate
S = two-point spatial correlation function as defined by Eq. (6)
t = time
T = temperature
r' = rms fluctuation velocity in turbulence field
V = scattering volume
x = downstream distance from hypersonic object
y_f = half-width of turbulent wake front
Z = charge on molecular ion, in units of electronic charge
$\Delta(\ )$ = deviation of ( ) quantity from its mean value
$\overline{(\ )}$ = mean value of ( ) quantity
d^3r = volume element in physical space
α = electric polarizability per molecule
β, γ = power law indices for spectrum function
Γ = scalar fluctuation spectrum
ϵ = dielectric constant
$\mathcal{E}$ = turbulence energy dissipation rate
η = ratio between correlation length and wake half-width
κ = extinction coefficient
λ_0 = wavelength of incident electromagnetic wave
Λ_u = velocity correlation length scale in turbulence field
Λ_ϵ = dielectric constant correlation length scale in turbulence field
ν = collision frequency
ρ = mass density
σ = scattering cross section
χ = electric susceptibility per unit volume
ψ = angle between scattered wave vector $\mathbf{k}$ and incident wave electric field vector $\mathbf{E}_0$
ω = radian frequency of electromagnetic wave
Ω, Ω_2 = squared frequency ratio as defined prior to Eq. (23)

Subscripts

0 = incident wave, or free space condition
e = for electron
i = for ion
j = member of statistical ensemble
ζ = for type ζ molecule
∞ = ambient condition
max = maximum value

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I. Introduction

RADAR scattering from large meteor trails and from the turbulent wake of other high-speed objects has been an interesting subject of study for many years.[1-3] Quantitative treatment of this problem requires not only adequate prescription of the chemical and statistical properties of the turbulent flowfield, but also an applicable theory of electromagnetic wave scattering for the particular plasma condition of interest. Although the mathematical problem of scattering from a statistically distributed dielectric medium, such as a turbulent plasma, is generally very complex,[4,5] it has been shown that under certain restrictive conditions regarding the plasma density and the over-all dimension of the scattering volume,[4] the averaged scattering intensity can be reliably calculated according to the Booker formula,[6] which was based on the first-order Born approximation[7] and requires only relatively simple specification of the statistical properties of the scattering medium. In particular, it has been shown that[4,6] when the "underdensed" turbulent plasma is isotropic and statistically homogeneous throughout the scattering volume, the radar backscattering intensity per unit scattering volume is directly proportional to the product of the mean-square dielectric constant fluctuation and a certain normalized "spectral function" evaluated at twice the incident wave number $k_0 \equiv 2\pi/\lambda_0$.

In the present paper, we shall follow the formal procedure in deriving such a spectral function for a turbulent wake plasma which is not necessarily isotropic nor statistically homogeneous within the scattering volume, and examine all the critical physical parameters that enter into the determination of this function. An attempt will also be made to estimate the approximate shape of such spectral function in the short-wave limit (i.e., near diffusive cutoff) for some typical far wake conditions.

II. Booker Formula

According to the Booker formula,[4,6] the instantaneous differential scattering cross section of monochromatic electromagnetic waves from a time-varying inhomogeneous dielectric medium occupying a certain scattering volume V is given by

$$\sigma(\theta,t) = \frac{k^4 \sin^2\psi}{4\pi} \left| \int_V \Delta\epsilon(\mathbf{r},t) e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3r \right|^2 \tag{1}$$

Here $\sigma(\theta,t)$ is defined, as usual, as 4π times the power scattered per unit solid angle per unit incident power density along the direction of scattering $\mathbf{k}$, which makes an angle θ with respect to the incident wave propagation vector $\mathbf{k}_0$; $k \equiv |\mathbf{k}| = |\mathbf{k}_0| \equiv 2\pi/\lambda_0$ is the incident wave number; ψ is the angle between $\mathbf{k}$ and the electric field vector $\mathbf{E}_0$ of the incident wave; $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}_0$ is a constant vector of magnitude $2k \sin(\theta/2)$ and oriented in a direction perpendicular to the angular bisector of $\mathbf{k}$ and $\mathbf{k}_0$;

$$\Delta\epsilon(\mathbf{r},t) \equiv \epsilon(\mathbf{r},t) - \bar{\epsilon} \tag{2}$$

is the deviation of the local dielectric constant at $(\mathbf{r},t)$ from some suitably taken spatial average of the dielectric constant within the scattering volume V at the instant of observation t. By letting $\mathbf{r}' \equiv \mathbf{r}_2 - \mathbf{r}_1$ be the positional vector difference between two arbitrary points 1 and 2 within the volume of integration, Eq. (1) can be rewritten as

$$\begin{aligned}\sigma(\theta,t) &= \frac{k^4 \sin^2\psi}{4\pi} \int_V \Delta\epsilon(\mathbf{r}_1,t) e^{-i\mathbf{q}\cdot\mathbf{r}_1}\, d^3r_1 \int \Delta\epsilon(\mathbf{r}_2,t) e^{i\mathbf{q}\cdot\mathbf{r}_2}\, d^3r_2 \\ &= \frac{k^4 \sin^2\psi}{4\pi} \int_V \int_V \Delta\epsilon(\mathbf{r}_1,t)\, \Delta\epsilon\,(\mathbf{r}_1 + \mathbf{r}',t) \times \\ &\qquad e^{i\mathbf{q}\cdot\mathbf{r}'}\, d^3r_1 d^3r' \end{aligned} \tag{1a}$$

When applied to a turbulent dielectric medium, it is important to note that an "instantaneous" scattering cross section for monochromatic waves exists only when the time scale for any significant change in spatial configuration of $\Delta\epsilon(\mathbf{r},t)$ is relatively long in comparison with the propagation time of electromagnetic wave over the longest dimension of the scattering volume V of interest. In the case of back scattering from a pulsed radar, the longest dimension of V is usually determined by the length of the range gate ΔR, so that the propagation time of interest is $2\Delta R/c$, with c being the velocity of light. The time scale for significant change of the turbulent medium, on the other hand, is given by Λ_u/u' where Λ_u is some characteristic length scale (e.g., integral scale of the two-point velocity correlation function), and u' is the rms fluctuation velocity of the turbulence field. Thus, the criterion of slowly changing scattering medium configuration is, in effect,

$$\Delta R/\Lambda_u \ll c/u' \tag{3}$$

Since c/u' is generally greater than 10^6 in any scattering field of subsonic turbulence velocity, the foregoing criterion is quite easily satisfied.

By treating t in (1a) as a statistical sampling parameter, one may then proceed to define an *averaged* differential scattering cross section $\sigma(\theta)$, such that

$$\begin{aligned}\sigma(\theta) &\equiv \frac{1}{N} \sum_{j=1}^{N} \sigma(\theta,t_j) \\ &= \frac{k^4 \sin^2\psi}{4\pi} \int_V \int_V \frac{1}{N} \sum_{j=1}^{N} \Delta\epsilon(\mathbf{r}_1,t_j) \times \\ &\qquad \Delta\epsilon(\mathbf{r}_1 + \mathbf{r}',t_j) e^{i\mathbf{q}\cdot\mathbf{r}'}\, d^3r_1 d^3r' \end{aligned} \tag{4}$$

where subscript j denotes the jth member of a statistical ensemble of N members ($N \gg 1$). One may further define a mean-square fluctuation intensity $\overline{\Delta\epsilon^2(\mathbf{r}_1)}$ and a normalized two-point autocorrelation function $S(\mathbf{r}_1,\mathbf{r}')$ for the local statistical fluctuation of dielectric constant,[4,6]

$$\overline{\Delta\epsilon^2(\mathbf{r}_1)} \equiv \frac{1}{N} \sum_{j=1}^{N} \Delta\epsilon(\mathbf{r}_1,t_j)\Delta\epsilon(\mathbf{r}_1,t_j) \tag{5}$$

$$S(\mathbf{r}_1,\mathbf{r}') \equiv \sum_{j=1}^{N} \Delta\epsilon(\mathbf{r}_1,t_j)\Delta\epsilon(\mathbf{r}_1 + \mathbf{r}',t_j) \Bigg/ \sum_{j=1}^{N} \Delta\epsilon(\mathbf{r}_1,t_j)\Delta\epsilon(\mathbf{r}_1,t_j) \tag{6}$$

In terms of these statistical averaged quantities, Eq. (4) becomes

$$\begin{aligned}\sigma(\theta) &= \frac{k^4 \sin^2\psi}{4\pi} \int_V \int_V \overline{\Delta\epsilon^2(\mathbf{r}_1)} S(\mathbf{r}_1,\mathbf{r}') e^{i\mathbf{q}\cdot\mathbf{r}'}\, d^3r_1 d^3r' \\ &= \frac{k^4 \sin^2\psi}{4\pi} \int_V \overline{\Delta\epsilon^2(\mathbf{r}_1)}\, F(\mathbf{r}_1,\mathbf{q}) d^3r_1 \end{aligned} \tag{7}$$

where

$$F(\mathbf{r}_1,\mathbf{q}) \equiv \int_V S(\mathbf{r}_1,\mathbf{r}') e^{i\mathbf{q}\cdot\mathbf{r}'}\, d^3r' \tag{8}$$

may be called the local spectrum function for scattering from dielectric constant fluctuations about a fixed point $\mathbf{r}_1$. It is a Fourier transform of $S(\mathbf{r}_1,\mathbf{r}')$ along the direction of $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}_0$, and is therefore a measure of the relative strength of distributed dielectric gradient length scale along a direction perpendicular to the angular bisector between the incident and scattered wave vectors. It may be noted that such a local spectrum function can be defined for any *given* statistical ensemble of turbulence field realizations, whether the statistical fluctuation of $\Delta\epsilon$ is homogeneous and isotropic or not. The function $F(\mathbf{r}_1, \mathbf{q})$ so defined will be insensitive to the boundary conditions as long as the dimensions of the scattering volume V are much greater than the correlation length scale Λ_ϵ for the dielectric constant fluctuations.

III. Relationship between $\Delta\epsilon$ and Plasma Density Fluctuation

The plasma generated in the wake of hypersonic objects in the earth's atmosphere is a partially ionized, multicomponent gas mixture consisting of neutral atoms, molecules, free electrons, as well as atomic and molecular ions of both positive and negative charges. Although the major chemical species are expected to derive from thermal decomposition and subsequent chemical reactions of the O_2-N_2 system,[8-10] ablation products and normal atmospheric contaminants such as Ar, CO_2, H_2O, Na, etc. may also contribute.

In view of the fact that severe initial temperature inhomogeneities in the hypersonic wake may lead to large amplitude mass density fluctuation,[9] one may expect that the instantaneous spatial distribution of dielectric constant $\epsilon(\mathbf{r},t)$ would depend on the instantaneous temperature and pressure distributions as well as on the distribution of free electrons and ions. We shall now proceed to derive an explicit relationship between $\Delta\epsilon$ and the fluctuations of these various spatial variables.

Assuming that the degree of ionization (i.e., fraction of air molecules ionized) in most wake plasmas[8-10] is sufficiently low to justify the neglect of collective plasma effects, one may determine the local dispersion properties of the gas mixture according to kinetic theory and considering only linear transport.[11] Even so, the index of refraction for an ionized gas mixture will still be a rather complicated function of electromagnetic wave frequency, field strength, gas temperature, density, and chemical composition. However, under the condition of weak field and sufficiently mild velocity dependence of the electron-molecule momentum transfer cross section, the Lorentz formula[11,12] appears to yield a reasonably good approximation to the complex electrical conductivity attributable to the motion of the free electrons,

$$K_e = (n_e e^2/m_e)[(\nu_e - i\omega)/(\nu_e^2 + \omega^2)] \tag{9}$$

where n_e denotes the local number density of free electrons; e and m_e are the electronic charge and mass; ω is 2π times the applied field frequency; and ν_e is some suitably defined total collision frequency between an averaged electron and all other particles in the gas mixture.[12]

The imaginary part of the complex conductivity due to the induced motion of the free electrons thus gives rise to an electric susceptibility per unit volume,[11]

$$\chi_e = -(n_e e^2/m_e)[1/(\nu_e^2 + \omega^2)] \tag{10}$$

Similarly, the induced motions of all the positive and negative ions give rise to a susceptibility per unit volume

$$\chi_i = -\sum_{\text{all ions}} \frac{n_\zeta (Z_\zeta e)^2}{m_\zeta} \frac{1}{\nu_\zeta^2 + \omega^2} \tag{11}$$

where $Z_\zeta e$ represents the net electric charge on an ion of chemical description ζ; and n_ζ, m_ζ, and ν_ζ denote, respectively, the number density, mass, and collision frequency for that particular chemical species.

The electric susceptibility due to the internal structures of all atoms, molecules, and ions are additive according to the Clausius-Mosotti relation,[13]

$$\chi_m = \sum_{\text{all particles}} n_\zeta \alpha_\zeta \tag{12}$$

Here, α_ζ represents the electric polarizability per particle of chemical description ζ, averaged over molecular orientation, and evaluated at the applied field frequency $\omega/2\pi$.

The dielectric constant of a medium in Gaussian units is defined as unity plus 4π times the total electric susceptibility per unit volume. Thus, from Eqs. (10-12), one obtains the following expression for the local dielectric constant for the weakly ionized plasma:

Table 1 Published values[14] of electric polarizability per molecule, averaged over orientation, in atomic units

Molecule	α/a_0^3	Atom	α/a_0^3
N_2	12.1	N	7.6
O_2	11.1	O	5.2
NO	12	Ar	11.1
CO	13.5	H	4.5
CO_2	17.3		
H_2O	10.0		

$$\begin{aligned}\epsilon &= 1 + 4\pi(\chi_e + \chi_i + \chi_m)\\ &= 1 - 4\pi\left[\frac{n_e e^2}{m_e}\frac{1}{\nu_e^2+\omega^2} + \sum_{\text{all ions}}\frac{n_\zeta(Z_\zeta e)^2}{m_\zeta}\frac{1}{\nu_\zeta^2+\omega^2} - \sum_{\text{all particles}} n_\zeta\alpha_\zeta\right]\end{aligned} \tag{13}$$

The polarizability α_ζ for all atoms, molecules, and ions at field frequencies far away from any resonance lines is practically independent of frequency. Typical published values of α_ζ for the major chemical species to be expected in hypersonic wakes are shown in Table 1. From these published values, it is seen that the polarizability per equivalent air molecule of normal composition is approximately $12a_0^3$, where $a_0 \equiv h^2/4\pi^2 m_e e^2 = 5.292 \times 10^{-9}$ cm is the atomic unit of length ($a_0 \equiv$ radius of the first Bohr orbit; $h = 6.626 \times 10^{-27}$ erg-sec $\equiv$ Planck's constant). Thus, by introducing an *equivalent number density of air molecules* $n_{\text{air}} \equiv \rho/m_{\text{air}}$, where $\rho \equiv \rho(\mathbf{r},t)$ represents the local mass density of the wake plasma, and $m_{\text{air}} = 4.83 \times 10^{-23}$ g is the averaged mass per air molecule, one may replace the last term in Eq. (13) by the approximation

$$\sum_{\text{all particles}} n_\zeta \alpha_\zeta \cong 12a_0^3 n_{\text{air}} = \frac{12a_0^3\rho}{m_{\text{air}}} \tag{14}$$

One may further introduce the atomic unit of frequency, i.e., the Rydberg frequency,

$$Ry \equiv e^2/2a_0 h = 3.290 \times 10^{15} \text{ sec}^{-1} \tag{15}$$

so that Eq. (13) becomes

$$\epsilon = 1 - (4\pi a_0)^3\left(\frac{Ry^2}{\nu_e^2+\omega^2}n_e + \sum_{\text{all ions}}\frac{m_e}{m_\zeta}\frac{Ry^2}{\nu_\zeta^2+\omega^2}Z_\zeta^2 n_\zeta\right) + \frac{48\pi a_0^3\rho}{m_{\text{air}}} \tag{16}$$

In the turbulent wake plasma, the electron and ion collision frequencies ν_e, ν_ζ are all functions of space and time so that the fluctuation in dielectric constant $\Delta\epsilon$ depends on the fluctuations of these quantities as well as on the fluctuations of n_e, n_ζ, and ρ. The above equation shows that a linear relationship between $\Delta\epsilon$ and Δn_e, Δn_ζ and $\Delta\rho$ exists only under either one of the following two restrictive conditions: 1) $\nu_e^2 \ll \omega^2$ (which also implies $\nu_\zeta^2 \ll \omega^2$ since $\nu_\zeta^2/\nu_e^2 \cong m_e/m_\zeta \ll 1$); 2) ν_e and ν_ζ are approximately constant.

It may be noted that in order for the Booker formula (1) to be valid, it was already implicitly assumed that the real part of the electrical conductivity [Eq. (9)]

$$\sigma_e = (n_e e^2/m_e)[\nu_e/(\nu_e^2 + \omega^2)]$$

contributes relatively little to the perturbation of the complex index of refraction $N \equiv n - i\kappa$. From the well known dispersion formula,[11]

$$\begin{aligned}n^2 &= (\epsilon/2)\{1 + [1 + (4\pi\sigma_e/\epsilon\omega)^2]^{1/2}\}\\ \kappa^2 &= (\epsilon/2)\{-1 + [1 + (4\pi\sigma_e/\epsilon\omega)^2]^{1/2}\}\end{aligned} \tag{17}$$

it is seen that weak perturbation from σ_e does not necessarily imply $\nu_e \ll \omega$ as one might have inferred directly from Eq. (9), but rather

$$(\pi n_e e^2/m_e\omega^2)[\nu_e^2/(\nu_e^2 + \omega^2)] \ll 1 \tag{18}$$

This condition can be satisfied for arbitrary values of ν_e/ω only when the plasma is very much *underdensed*, i.e.,

$$n_e \ll n_e^* \equiv m_e\omega^2/4\pi e^2 \tag{19}$$

though $\nu_e^2 \ll \omega^2$ would also be a sufficient condition as long as n_e is not large in comparison with n_e^*.

In a spirit consistent with that in the derivation of the Lorentz formula (9), the total electron collision frequency ν_e can be expressed as a simple sum of partial collision frequencies with the various species ζ in the dissociated gas mixture, such that

$$\nu_e = \bar{C}_e \sum_{\text{all } \zeta \neq e} n_\zeta \bar{Q}_\zeta \tag{20}$$

where $\bar{C}_e \equiv (8kT/\pi m_e)^{1/2}$ is the mean thermal speed of the free electrons at the local temperature T, and $\bar{Q}_\zeta$ is the averaged momentum transfer cross section between the electrons and the type ζ particles.

From previous extensive studies[15–22] of the low-energy electron scattering properties of the major chemical species which make up a typical hypersonic wake plasma,[8,9] one may conclude that the atomic constituents O, N, Ar are not nearly as important as the molecular constituents N_2, O_2, NO in determining ν_e and that the absolute value of $\bar{Q}_\zeta$ for all these species in the temperature range of interest are no longer much in doubt. Even though the exact energy dependence of Q_ζ for such major species as N_2 at very low electron energies (less than 0.3 ev) still appears somewhat controversial,[16,17] existing experimental data point overwhelmingly toward a mixture-averaged cross section $\bar{Q} \equiv \Sigma n_\zeta \bar{Q}_\zeta / \Sigma n_\zeta$ which is either nearly constant at $10\pi a_0^2$ in the temperature range $300 \leq T \leq 6000°K$, or with a mild temperature dependence like $T^{1/2}$. In a turbulent wake plasma of subsonic turbulence velocity $u' < a$ [where $a \equiv (\gamma p/\rho)^{1/2}$ denotes the local speed of sound], the fractional pressure fluctuation $p'/\bar{p}$ caused by inertial forces[23] is likely to be much weaker than the fractional temperature fluctuation $T'/\bar{T}$ caused by initial entropy inhomogeneities in the flowfield,[9] so that the local gas density and temperature are nearly perfectly anticorrelated according to the perfect gas law

$$p = kT \sum_\zeta n_\zeta \cong \text{const} \equiv \bar{p} \tag{21}$$

(Here we are assuming that no important temperature relaxation effects are present, so that all particles share the same local translational temperature T.) Equations (20) and (21), together with our preceding conclusion about the mixture-averaged cross section Q, then lead to a total electron collision frequency which either depends weakly on the local temperature

$$\nu_e \cong 10\pi a_0^2(8/\pi m_e kT)^{1/2}\bar{p} \tag{22a}$$

or does not depend on the local temperature at all,

$$\nu_e = \bar{Q}(\bar{T})(8/\pi m_e k\bar{T})^{1/2}\bar{p} \cong 7 \times 10^{10}(\bar{p}/p_0) \text{ sec}^{-1} \tag{22b}$$

This last expression indicates that at normal sea-level atmospheric pressure p_0, the electron collision frequency ν_e is comparable to the wave frequency ω for microwave radars, but at the higher altitudes corresponding to the onset of turbulent wake formation,[3] the point at which $\omega = \nu_e$ tends to shift toward the lower end of the radar frequency scale.

By treating ν_e as only a function of ambient pressure $\bar{p}$ [i.e., Eq. (22b)] and neglecting ν_ζ in Eq. (16), one finally obtains, with the aid of the abbreviating symbols, $\Omega_e \equiv Ry^2/(\nu_e^2 + \omega^2)$, $\Omega \equiv Ry^2/\omega^2$, the following expressions for the dielectric constant fluctuation and its two-point correlation function:

$$\Delta\epsilon = -(4\pi a_0)^3\left[\Omega_e\Delta n_e + \Omega\sum_\zeta \frac{m_e}{m_\zeta} Z_\zeta^2\Delta n_\zeta - \frac{3}{4\pi^2 m_{\text{air}}}\Delta\rho\right] \tag{23}$$

$$\begin{aligned}\overline{\Delta\epsilon(\mathbf{r}_1)\Delta\epsilon(\mathbf{r}_1+\mathbf{r}')} &\equiv \frac{1}{N}\sum_{j=1}^{N}\Delta\epsilon(\mathbf{r}_1,t_j)\Delta\epsilon(\mathbf{r}_1+\mathbf{r}',t_j) = \\ &(4\pi a_0)^6\left\{\Omega_e^2\overline{\Delta n_e(\mathbf{r}_1)\Delta n_e(\mathbf{r}_1+\mathbf{r}')} + \right. \\ &\Omega_e\Omega\sum_\zeta \frac{m_e}{m_\zeta}Z_\zeta^2[\overline{\Delta n_e(\mathbf{r}_1)\Delta n_\zeta(\mathbf{r}_1+\mathbf{r}')} + \\ &\overline{\Delta n_e(\mathbf{r}_1+\mathbf{r}')\Delta n_\zeta(\mathbf{r}_1)}] - \frac{3\Omega}{4\pi^2 m_{\text{air}}} \times \\ &[\overline{\Delta n_e(\mathbf{r}_1)\Delta\rho(\mathbf{r}_1+\mathbf{r}')} + \overline{\Delta n_e(\mathbf{r}_1+\mathbf{r}')\Delta\rho(\mathbf{r}_1)}] - \\ &\frac{3\Omega}{4\pi^2 m_{\text{air}}}\sum_\zeta\frac{m_e}{m_\zeta}Z_\zeta^2[\overline{\Delta n_\zeta(\mathbf{r}_1)\Delta\rho(\mathbf{r}_1+\mathbf{r}')} + \\ &\overline{\Delta n_\zeta(\mathbf{r}_1+\mathbf{r}')\Delta\rho(\mathbf{r}_1)}] + \Omega^2\sum_\zeta\sum_{\zeta'}\frac{m_e^2}{m_\zeta m_{\zeta'}} \times \\ &Z_\zeta^2 Z_{\zeta'}^2\,\overline{\Delta n_\zeta(\mathbf{r}_1)\Delta n_{\zeta'}(\mathbf{r}_1+\mathbf{r}')} + \\ &\left.\left(\frac{3}{4\pi^2 m_{\text{air}}}\right)^2\overline{\Delta\rho(\mathbf{r}_1)\Delta\rho(\mathbf{r}_1+\mathbf{r}')}\right\}\end{aligned} \tag{24}$$

Substitution of (24) into Eq. (4) then yields a resultant scattering cross section consisted of six distinct terms, corresponding to the six different correlations,

$$\sigma(\theta) = \sigma_{ee}(\theta) + \sigma_{ei}(\theta) + \sigma_{ep}(\theta) + \sigma_{ip}(\theta) + \sigma_{ii}(\theta) + \sigma_{pp}(\theta) \tag{25}$$

where

$$\sigma_{ee}(\theta) = \frac{k^4\sin^2\psi}{4\pi}(4\pi a_0)^6\Omega_e^2 \times \int_V\int_V \overline{\Delta n_e^2(\mathbf{r}_1)}S_{ee}(\mathbf{r}_1,\mathbf{r}')e^{i\mathbf{q}\cdot\mathbf{r}'}d^3r_1d^3r' \tag{26}$$

$$\sigma_{ei}(\theta) = \frac{k^4\sin^2\psi}{4\pi}(4\pi a_0)^6\Omega_e\Omega\cdot\times \sum_\zeta\frac{m_e}{m_\zeta}Z_\zeta^2\int_V\int_V[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}[\overline{\Delta n_\zeta^2(\mathbf{r}_1)}]^{1/2}\times S_{e\zeta}(\mathbf{r}_1,\mathbf{r}')e^{i\mathbf{q}\cdot\mathbf{r}'}d^3r_1d^3r' \tag{27}$$

$$\sigma_{ep}(\theta) = -\frac{k^4\sin^2\psi}{4\pi}(4\pi a_0)^6\frac{3\Omega_e}{4\pi^2 m_{\text{air}}}\times \int_V\int_V[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}[\overline{\Delta\rho^2(\mathbf{r}_1)}]^{1/2}S_{ep}(\mathbf{r}_1,\mathbf{r}')e^{i\mathbf{q}\cdot\mathbf{r}'}d^3r_1d^3r' \tag{28}$$

and similar expressions for other terms. The *normalized* autocorrelation function for the free electrons $S_{ee}(\mathbf{r}_1,\mathbf{r}')$ is defined in the same manner as that for $\Delta\epsilon$ in Eq. (6), whereas the normalized cross-correlation functions $S_{e\zeta}(\mathbf{r}_1,\mathbf{r}')$ and $S_{ep}(\mathbf{r}_1,\mathbf{r}')$ are defined here by

$$S_{e\zeta}(\mathbf{r}_1,\mathbf{r}') \equiv [\overline{\Delta n_e(\mathbf{r}_1)\Delta n^\zeta(\mathbf{r}_1+\mathbf{r}')} + \overline{\Delta n_e(\mathbf{r}_1+\mathbf{r}')\Delta n_\zeta(\mathbf{r}_1)}]/[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}[\overline{\Delta n_\zeta^2(\mathbf{r}_1)}]^{1/2} \tag{29}$$

$$S_{ep}(\mathbf{r}_1,\mathbf{r}') \equiv [\overline{\Delta n_e(\mathbf{r}_1)\Delta\rho(\mathbf{r}_1+\mathbf{r}')} + \overline{\Delta n_e(\mathbf{r}_1+\mathbf{r}')\Delta\rho(\mathbf{r}_1)}]/[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}[\overline{\Delta\rho^2(\mathbf{r}_1)}]^{1/2} \tag{30}$$

IV. Relative Importance of the Different Correlation Terms

Equation (24) shows that the autocorrelation function for dielectric constant fluctuation depends not only on the autocorrelation of free electrons, but also on the cross correlations among the free electrons, the molecular ions, and the mass density, as well as on the autocorrelation of the latter two. Even though neither direct experimental measurement nor predictive theory yet exists to help one determine the relative magnitude of these cross correlations among the different thermodynamic variables in *any* kind of turbulent plasma flow, elementary theoretical considerations do point toward the existence of strong spatial correlation between the local number density of free electrons and those of the molecular ions, as well as strong spatial anticorrelation between the number density of free electrons and the local mass density in a typical turbulent wake plasma. The reasoning is as follows:

The existence of strong spatial anticorrelation between the free electrons and the mass density is a direct consequence of the fact that all the free electrons were initially generated upstream from the high-temperature regions of the hypersonic flowfield[9] (i.e., strong shock wave and/or boundary layers), and that temperature and mass density in the turbulent wake are expected to be strongly anticorrelated [Eq. (21)]. The tendency toward strong spatial correlation between the free electrons and the molecular ions can be deduced not only from the local charge-neutrality condition (i.e., ambipolar diffusion[24]), but also from the fact that negative ion formation through the attachment processes[9,10] can take place only in regions where the electrophilic molecules can find free electrons. The local charge-neutrality argument is valid, of course, only when the local Debye length for the free electrons

$$l_{De} \equiv (kT/4\pi n_e e^2)^{1/2} \tag{31}$$

remains smaller than the correlation length scale of interest. In the case of radar back scattering at incident wavelength λ_0, the correlation length scale of interest is $q^{-1} = \lambda_0/4\pi$ [see Eq. (4)], so that the condition of sufficiently small

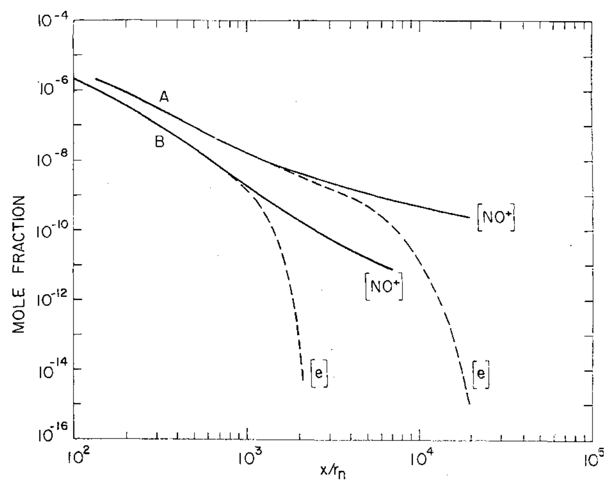

Fig. 1 Numerical example of electron mole fraction $[e] = \bar{n}_e/\bar{n}_{air}$ and positive ion mole fraction $[NO^+] = \bar{n}_{NO^+}/\bar{n}_{air}$ in the wake of hypersonic spheres as functions of the normalized axial distance x/r_n. Example A: sphere of radius $r_n = 0.238$ cm, $U_\infty = 18{,}800$ ft/sec in room temperature air at 40 mm Hg pressure (Ref. 10). Example B: $r_n = 1$ ft, $U_\infty = 22{,}000$ ft/sec at 100,000-ft altitude in standard atmosphere.[9] Note that the total mole fraction for all negative ions is given by the charge neutrality condition $\Sigma[-\text{ions}] = [NO^+] - [e]$.

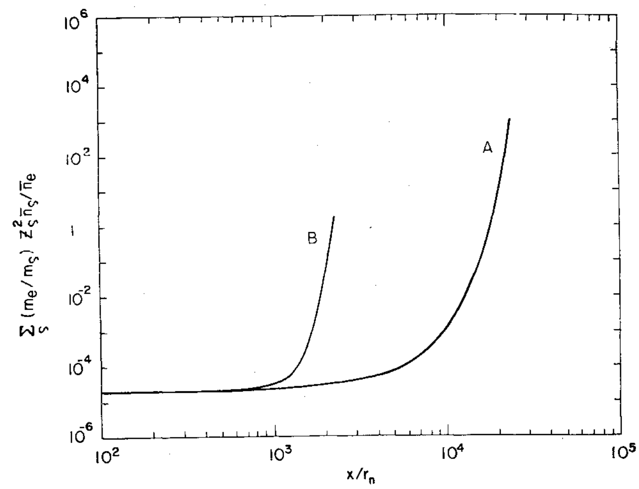

Fig. 2 Parameter for measuring the relative magnitude between the ion-electron and the electron-electron correlation terms in Eq. (24), as computed from the two numerical examples cited in Fig. 1.

Debye length is satisfied if (with T in °K and λ_0 in cm)

$$n_e \gg 4\pi kT/e^2\lambda_0^2 \cong 8 \times 10^3 T/\lambda_0^2 \text{ electrons/cm}^3 \tag{32}$$

A strong spatial correlation between Δn_ζ and Δn_e implies that the ratio $\overline{\Delta n_e(\mathbf{r}_1)\Delta n_\zeta(\mathbf{r}_1 + \mathbf{r}')}/\overline{\Delta n_e(\mathbf{r}_1)\Delta n_e(\mathbf{r}_1 + \mathbf{r}')}$ is nearly the same as the ratio between their respective mean absolute fluctuations $\overline{|\Delta n_\zeta(\mathbf{r}_1)|}/\overline{|\Delta n_e(\mathbf{r}_1)|}$. Since this latter ratio is also comparable to the ratio between their rms fluctuations $[\overline{\Delta n_\zeta^2(\mathbf{r}_1)}]^{1/2}/[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}$, this is equivalent to the statement that the normalized correlation functions $S_{e\zeta}(\mathbf{r}_1,\mathbf{r}')$ and $S_{ee}(\mathbf{r}_1,\mathbf{r}')$ are about the same. Consequently, comparison between Eqs. (26) and (27) shows that the relative contributions to the local scattering intensity per unit volume from the electron-ion and electron-electron correlations bear the ratio

$$\frac{\sigma_{ei}(\theta,\mathbf{r}_1)}{\sigma_{ee}(\theta,\mathbf{r}_1)} \cong \frac{\Omega}{\Omega_e} \sum_\zeta \frac{m_e}{m_\zeta} \frac{Z_\zeta^2[\overline{\Delta n_\zeta^2(\mathbf{r}_1)}]^{1/2}}{[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}}$$

$$\cong \frac{\Omega}{\Omega_e} \sum_\zeta \frac{m_e}{m_\zeta} \frac{Z_\zeta^2 \overline{n_\zeta(\mathbf{r}_1)}}{\overline{n_e(\mathbf{r}_1)}} \tag{33}$$

The second near-equality statement follows from the fact that the Δn_ζ and Δn_e represent fluctuations of two different chemical species mixed by the same turbulence field, so that their relative fluctuation intensities $[\overline{\Delta n_\zeta^2(\mathbf{r}_1)}]^{1/2}/\overline{n_\zeta(\mathbf{r}_1)}$ and $[\overline{\Delta n_e^2(\mathbf{r}_1)}]^{1/2}/\overline{n_e(\mathbf{r}_1)}$ are expected to be of roughly the same magnitude.

In view of the fact that $\Omega/\Omega_e \geq 1$, and Z_ζ^2 is either 1, 4, 9, or other squared integer for all positive and negative ions, the result of (33) shows that the electron-ion correlations will become of comparable importance to the electron-electron correlation when the total ion/electron concentration ratio approaches the averaged ion/electron mass ratio. The fact that such high ion/electron concentration ratio is often reached in the far wake of hypersonic objects is illustrated in Figs. 1 and 2, where the electron mole fraction $[e] \equiv \bar{n}_e/\bar{n}_{air}$, positive ion mole fraction $[NO^+] \equiv \bar{n}_{NO^+}/\bar{n}_{air}$, and the quantity $\Sigma(m_e/m_\zeta)Z_\zeta^2\bar{n}_\zeta/\bar{n}_e$ are plotted as functions of the normalized axial distance x/r_n from two numerical examples. Example A, taken from the recent calculations of E. A. Sutton for a small sphere of radius $r_n = 0.238$ cm travelling at 18,800-ft/sec velocity in room temperature air of 40 mmHg, is typical of conditions encountered in ballistic range experiments, whereas example B, taken from the earlier calculations of Lin and Hayes[9] for a sphere of 1-ft radius travelling at 22,000-ft/sec velocity and 100,000-ft standard altitude, is

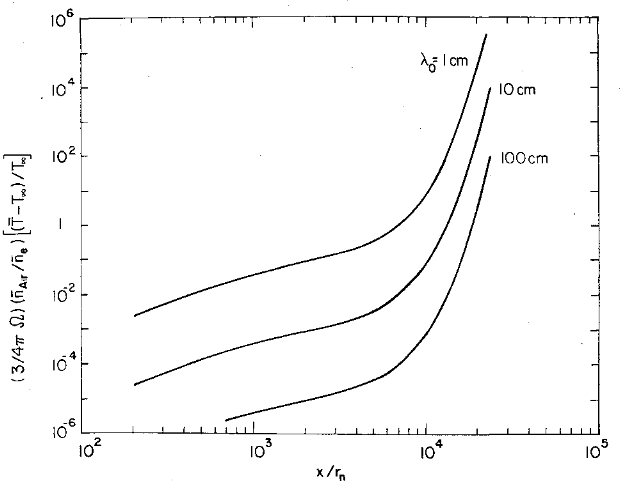

Fig. 3 Parameter for measuring the relative magnitude between the electron-mass density and the electron-electron correlation terms in Eq. (24), as computed from the electron mole fraction illustrated in Fig. 1 for example A, and from mean temperature defect taken from Refs. 9 and 10.

more typical of spacecraft re-entry. It should be noted, however, that in the earlier calculations of Lin and Hayes,[9] a relatively crude electron chemistry model involving only a single attachment reaction

$$e + O_2 + O_2 \rightleftarrows O_2^- + O_2 \tag{34}$$

was employed so that the early onset of electron attachment in example B was no doubt very much exaggerated due to the neglect of such important redetachment reactions as[10]

$$O + O_2^- \rightarrow O_3 + e \tag{35}$$

$$O + O^- \rightarrow O_2 + e \tag{36}$$

It may further be noted that the assumption of "extremely rapid turbulent mixing" used in both examples[9,10] also tended to exaggerate the rate of disappearance of free electrons in the wake plasma, but such inaccuracies could only change the point of onset of attachment and not the exponentially diverging tendency between the electron and ion mole fractions thereafter.

Similar assessment of the relative importance of cross correlation between electron number density fluctuation and mass density fluctuation can be made by noting that a strong spatial anticorrelation between the two implies that $S_{ep}(\mathbf{r}_1,\mathbf{r}') \cong -S_{ee}(\mathbf{r}_1,\mathbf{r}')$. Thus, comparison between Eqs. (26) and (28) yields

$$\sigma_{ep}(\theta,\mathbf{r}_1)/\sigma_{ee}(\theta,\mathbf{r}_1) \cong (3/4\pi^2\Omega_e m_{\text{air}}) \times \overline{[\Delta\rho^2(\mathbf{r}_1)]}^{1/2}/\overline{[\Delta n_e^2(\mathbf{r}_1)]}^{1/2} \tag{37}$$

The rms mass density fluctuation is not directly related to the mean mass density $\bar{\rho}$ within the wake, but rather to the difference between $\bar{\rho}$ and the ambient air density ρ_∞, such that $\overline{[\Delta\rho^2(\mathbf{r}_1)]}^{1/2}/[\rho_\infty - \overline{\rho(\mathbf{r}_1)}]$ is a slowly varying quantity of magnitude somewhat smaller than unity.[27] If one now let

$$I_{\rho e}(\mathbf{r}_1) \equiv \frac{\overline{[\Delta\rho^2(\mathbf{r}_1)]}^{1/2}/[\rho_\infty - \overline{\rho(\mathbf{r}_1)}]}{\overline{[\Delta n_e^2(\mathbf{r}_1)]}^{1/2}/\overline{n_e(\mathbf{r}_1)}} \tag{38}$$

be another slowly varying quantity of order of magnitude unity which measures the relative intensity between mass density and electron density fluctuations, substitution of (38) into (37) then yields, with the aid of Eq. (21) and $\bar{n}_{\text{air}} \equiv \rho/m_{\text{air}}$,

$$\sigma_{ep}(\theta,\mathbf{r}_1)/\sigma_{ee}(\theta,\mathbf{r}_1) \cong (3/4\pi^2\Omega_e) \times [\overline{n_{\text{air}}(\mathbf{r}_1)}/\overline{n_e(\mathbf{r}_1)}]\{[\bar{T}(\mathbf{r}_1) - T_\infty]/T_\infty\}I_{\rho e}(\mathbf{r}_1) \tag{39}$$

Even though the value of $I_{\rho e}(\mathbf{r}_1)$ is somewhat uncertain,[27–29] the preceding formula shows that at a typical microwave frequency $\omega = 6\pi \times 10^{10}\ \text{sec}^{-1}$ (corresponding to $\lambda_0 = 1$ cm), $\Omega_e \leq 3 \times 10^8$, the local scattering intensity due to cross correlation between electron and mass density fluctuations will become greater than that due to electron-electron autocorrelation when the mean electron mole fraction $\overline{n_e(\mathbf{r}_1)}/\overline{n_{\text{air}}(\mathbf{r}_1)}$ drops below a level of the order of 10^{-10}.

To illustrate the relative magnitude σ_{ep}/σ_{ee} in a typical hypersonic wake, the parameter $(3/4\pi^2\Omega)[\bar{n}_{\text{air}}/\bar{n}_e][(\bar{T} - T_\infty)/T_\infty]$ corresponding to the electron mole fraction illustrated in example A of Fig. 1 and the mean wake temperature $\bar{T}$ calculated in Refs. 9 and 10 is plotted in Fig. 3 as a function of x/r_u for several fixed values of λ_0.

By repeating the same argument, one can readily show that the relative contributions to the local scattering intensity per unit volume from the three remaining correlation terms on the right side of Eq. (24) are roughly governed by the following ratios:

$$\frac{\sigma_{i\rho}(\theta,\mathbf{r}_1)}{\sigma_{ee}(\theta,\mathbf{r}_1)} \lesssim \frac{3\Omega}{4\pi^2\Omega_e^2}\left[\sum_\zeta \frac{m_e}{m_\zeta}\frac{Z_\zeta^2\overline{n_\zeta(\mathbf{r}_1)}}{\overline{n_e(\mathbf{r}_1)}}\right] \times \left[\frac{\overline{n_{\text{air}}(\mathbf{r}_1)}}{\overline{n_e(\mathbf{r}_1)}}\right]\left[\frac{(\bar{T} - T_\infty)}{T_\infty}\right]I_{\rho e}(\mathbf{r}_1) \tag{40}$$

$$\frac{\sigma_{ii}(\theta,\mathbf{r}_1)}{\sigma_{ee}(\theta,\mathbf{r}_1)} \cong \left[\frac{\Omega}{\Omega_e}\sum_\zeta \frac{m_e}{m_\zeta}\frac{Z_\zeta^2\overline{n_\zeta(\mathbf{r}_1)}}{\overline{n_e(\mathbf{r}_1)}}\right]^2 \tag{41}$$

$$\sigma_{\rho\rho}(\theta,\mathbf{r}_1)/\sigma_{ee}(\theta,\mathbf{r}_1) \cong \{(3/4\pi^2\Omega_e) \times [\overline{n_{\text{air}}(\mathbf{r}_1)}/\overline{n_e(\mathbf{r}_1)}][(\bar{T} - T_\infty)/T_\infty]I_{\rho e}(\mathbf{r}_1)\}^2 \tag{42}$$

V. Discussion

In the study of electromagnetic wave scattering from the ionosphere[6] and from other weakly ionized turbulent plasmas,[4,5] it has been a common practice to represent the dielectric constant (or refractive index) fluctuation by a single term corresponding to the fluctuation of free electron number density. From the foregoing comparison of the relative magnitudes among the different correlation terms shown on the right side of Eq. (24) and the numerical examples illustrated in Figs. 1, 2, and 3, it is quite clear that in the study of hypersonic wake plasma such a simple representation of the dielectric constant fluctuation may not always be adequate. The need to include other correlation terms in Eq. (24) will no doubt complicate the problem of spectrum function determination in accordance with Eqs. (7) and (8). The extent of this complication can be assessed as follows.

Referring to the examples illustrated in Fig. 1, the ionization decay history in a turbulent hypersonic wake flow can quite generally be divided into two distinct periods. The first is a recombination (or charge neutralization) dominated period[9,26] in which the electron and positive-ion mole fractions are roughly identical, and both decreases according to some negative power of the axial distance x. The second is an attachment dominated period in which the electron mole fraction decreases nearly exponentially with x, while the positive- and negative-ion mole fractions continue to decrease according to some weak power law with x. The point of transition between the two periods is known to be a sensitive function of the ambient air density,[9] electron chemistry,[10] and turbulent mixing rate.[9,28]

On account of the smallness of the electron/ion mass ratio, it is quite obvious from Eqs. (33) and (41) that the electron-ion and ion-ion correlation terms will be unimportant throughout the recombination-dominated period. From the exponentiating nature of the parameter common to the right side of Eqs. (33) and (41) after the onset of attachment (Fig. 2), and the fact that $\sigma_{ii}/\sigma_{ee} \cong (\sigma_{ei}/\sigma_{ee})^2$, it is also obvious that the electron-ion correlation terms will be important only within a certain narrow range of x about the crossover point x_{ei} where $\Sigma(m_e/m_\zeta)Z_\zeta^2\bar{n}_\zeta/\bar{n}_e = 1$, but the ion-ion correlation terms will dominate over both electron-electron and electron-ion correlations for all x beyond x_{ei}.

The relative magnitudes of the electron-mass density correlation term and of the mass density autocorrelation term [Eqs. (39) and (42), and Fig. 3], on the other hand, are sensitive to the incident wave frequency as well as to the electron mole fraction, so that the crossover point $x_{e\rho}$ where $\sigma_{e\rho} = \sigma_{ee}$ may lie either in the recombination-dominated region or in the attachment-dominated region.

Even under the ideal condition where the second crossover point $x_{e\rho}$ also happened to lie within the attachment-dominated region, the theoretical problem of spectrum function prediction [i.e., in the absence of an actual statistical ensemble of N physical realizations postulated in the formal derivation of $F(\mathbf{r}_1,\mathbf{q})$ in Eqs. (7) and (8)] in a recombination-dominated region where the dielectric constant fluctuation [Eq. (24)] is again reduced to a single term corresponding to electron density fluctuation is still formidable. The uncertainties here lie mainly in the dynamics and scalar mixing mechanism in a chemically active turbulent shear flow[9,10,28] where all the fluid properties (e.g., temperature, mass density, viscosity, molecular diffusivities, etc.) are spatial variables. In the crossover region between $x_{e\rho}$ and x_{ei}, the aforementioned uncertainties are, of course, further compounded by those associated with the strength of cross correlation among the electron-, ion-, and mass-density fluctuations.

In the far wake region beyond both crossover points $x_{e\rho}$ and x_{ei}, the dielectric constant fluctuation will be dominated only by terms corresponding to ion-ion correlations and to mass-density autocorrelation. Here the problem of spectrum function determination becomes much easier since the rates of chemical reactions and the amplitude of temperature (or density) fluctuation would have slowed down and subsided to such low levels that existing passive scalar mixing theories[30–35] and experiments[36–39] can be fully utililized. In particular, for the diffusion of molecular ions and of translational temperature in the gaseous wake mixture under consideration, the problem reduces to that of passive scalar mixing in a constant-property fluid with nearly unity Prandtl number. Since existing experiments[36–39] showed that the small-scale structure of the turbulence field in all fully developed wakes and jets of sufficiently high Reynolds number tend to be locally isotropic and homogeneous, this immediately leads to a general prediction that the local "intensity spectra" for ion- and mass-density fluctuations at high wave numbers must also be isotropic and of nearly identical shape to the energy spectrum for velocity fluctuation. The local "intensity spectrum" $\Gamma(\mathbf{r}_1,q)$ for fluctuation of any passive scalar quantity ϵ, and the energy spectrum $E(\mathbf{r}_1,q)$ for velocity fluctuation, in an isotropic turbulence field are, respectively, defined according to the equations

$$\overline{\Delta\epsilon^2(\mathbf{r}_1)} \equiv \int_0^\infty \Gamma(\mathbf{r}_1,q)dq \tag{43}$$

$$\frac{3}{2}\,\overline{u'^2(\mathbf{r}_1)} \equiv \int_0^\infty E(\mathbf{r}_1,q)dq \tag{44}$$

where $u'(\mathbf{r}_1)$ is one of the three components of local velocity fluctuation. A similar shape between the two spectra implies

$$\Gamma(\mathbf{r}_1,q)/\overline{\Delta\epsilon^2(\mathbf{r}_1)} = E(\mathbf{r}_1,q)/\tfrac{3}{2}\overline{u'^2(\mathbf{r}_1)} \tag{45}$$

From low-speed wake experiments,[36] the energy spectrum $E(\mathbf{r}_1,q)$ near the axis of the wake is found to be generally of the universal equilibrium form, namely, a $q^{-5/3}$ dependence within the inertial subrange $\Lambda_u^{-1} < q < q_\kappa/2\pi$, and a rapid

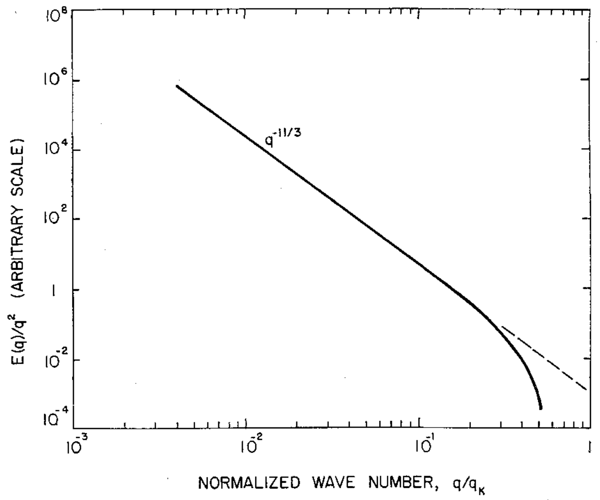

Fig. 4 Small-scale structure of the spectrum function F ($\mathbf{r}_1$,q) for dielectric constant fluctuation in the far wake as predicted from passive scalar mixing theories (Refs. 30–35) and universal velocity spectrum observed in low-speed wakes. Solid curve corresponds to mean value of $E(q)$ from the experimental points plotted in Fig. 10 of Ref. 36.

falloff beyond $q_\kappa/2\pi$. Here

$$q_\kappa \equiv (\varepsilon/\nu^3)^{1/4} \tag{46}$$

is the Kolmogorov wave number determined according to the local energy dissipation rate ε, and the local kinematic viscosity ν.

The intensity spectrum $\Gamma(\mathbf{r}_1,q)$ is related to the isotropic form of the spectrum function $F(\mathbf{r}_1,q)$ defined in Eq. (8) by[32]

$$2\pi^2\Gamma(\mathbf{r}_1,q)/q^2 = \overline{\Delta\epsilon^2(\mathbf{r}_1)}F(\mathbf{r}_1,q) \tag{47}$$

It follows that the high wave number portion of the latter spectrum function can also be expected to be of the universal form

$$F(\mathbf{r}_1,q) = [4\pi^2/3\overline{u'^2(\mathbf{r}_1)}]E(\mathbf{r}_1,q)/q^2 \tag{48}$$

A plot of this spectrum function according to the mean experimental data shown in Fig. 10 of Ref. 36 is illustrated here in Fig. 4. The Kolmogorov wave number q_κ needed for the scaling in a far wake can readily be estimated from the decay law of the turbulence velocity field in low-speed wakes.[36] An example of such estimate can be found in Fig. 7 of Ref. 28 (Pt. II).

As noted earlier, the electron density fluctuation spectrum in the far wake is generally more difficult to predict on account of uncertainties in the turbulent mixing mechanism associated with rapid chemical reactions (e.g., attachment) and with an electron diffusivity which may vary rapidly with local concentration (e.g., transition from ambipolar to free diffusion[24]). However, if the recombination-dominated region does extend itself to the very far wake, then diffusion of the ion pairs is likely to remain ambipolar down to the length scale corresponding to viscous cutoff $l_\kappa = 2\pi/q_\kappa$ [see Eqs. (31) and (32)]. Since the Prandtl number corresponding to ambipolar diffusion in a weakly ionized, single-temperature plasma is also of the order of unity, one may again expect the small-scale structure of the dielectric-constant fluctuation spectrum in such a recombination-dominated far wake to assume the universal form (48).

Although much can be said about the small-scale structure, the low wave number portion of the spectrum function cannot be so well generalized. As in any turbulent shear flow, the large scale structure of the scalar fluctuation spectrum can be expected to be anisotropic and depend rather strongly on initial conditions.

Table 2 Maximum radar scattering cross section per unit length due to ion-density fluctuations only in a hypersonic far wake depicted in example B of Fig. 1

x	meter	500	1000	2000	4000
$(d\sigma_{ii}/dx)_{\max}$	meter²/meter	6×10^{-9}	7×10^{-10}	9×10^{-11}	3×10^{-11}

Although much emphasis has been placed in pointing out the *relative* importance of molecular ions and mass-density fluctuations in governing the dielectric constant scattering spectrum in an attachment-dominated hypersonic far wake, a balancing remark concerning the *absolute* magnitude of the scattering intensity from such a weak plasma appears in order.

Considering only contributions from the molecular ions and back scattering (i.e., $\theta = \pi$, $\psi = \pi/2$, and $\mathbf{q} = -2\mathbf{k}_0$), we have, from Eqs. (7) and (23),

$$\sigma_{ii} = (4\pi)^5 \left(\frac{Ry}{c}\right)^4 a_0^6 \int_V \left[\sum_\zeta \frac{m_e}{m_\zeta} Z_\zeta^2 \Delta n_\zeta\right]^2 F(\mathbf{r}_1, -2\mathbf{k}_0) d^3 r_1 \tag{49}$$

The back-scattering cross section $\sigma_{ii}(\pi)$ thus depends on the incident wavelength only through the magnitude of the incident wave vector, $k_0 = 2\pi/\lambda_0$. It follows that the local (statistically-averaged) scattering intensity per unit volume would be a maximum when $2k_0$ is so chosen as to coincide with the wavenumber q^* where $F(\mathbf{r}_1, \mathbf{q})$ has a spectral maximum along the direction of $\mathbf{k}_0$. If the correlation function $S(\mathbf{r}_1, \mathbf{r}')$ were locally isotropic about $\mathbf{r}_1$ and dominated by a single scalar fluctuation, the spectrum function below viscous cutoff (i.e., $q < q_\kappa$) could quite often be represented by a simple function of the form[4,6]

$$F(\mathbf{r}_1, q) = A(q^2\Lambda_\epsilon^2)^\beta (1 + q^2\Lambda_\epsilon^2)^{-(\beta+\gamma)} \tag{50}$$

where β and γ are non-negative indices governing the asymptotic behavior of F in the low and high wave number regions, respectively, and A is a constant to be determined by the normalizing condition [from Eqs. (43) and (47)],

$$\int_0^\infty F(\mathbf{r}_1, q) q^2 dq = 2\pi^2 \tag{51}$$

This form of F has a spectral maximum at $q^*\Lambda_\epsilon = (\beta/\gamma)^{1/2}$ of magnitude

$$F_{\max} = 2\pi^2 \Lambda_\epsilon^3 g(\beta,\gamma) \tag{52}$$

where

$$g(\beta,\gamma) \equiv \left(\frac{\beta}{\gamma}\right)^\beta \left[1 + \frac{\beta}{\gamma}\right]^{-(\beta+\gamma)} \times \left[\int_0^\infty y^{2(\beta+1)} (1 + y^2)^{-(\beta+\gamma)} dy\right]^{-1} \tag{53}$$

is a dimensionless form factor of the order of unity over a wide range of possible values for the indices β and γ. The same form of F can also be modified on occasion to represent the scattering spectrum for a nonisotropic turbulence field with more than one correlation length scale.[6] In the case of hypersonic wake flow, the extent of anisotropy as well as the spectral form is uncertain, and the correlation length scales are expected to vary with axial distance x behind the object. Nevertheless, for the purpose of making order-of-magnitude estimate of the maximum scattering cross section per unit length of the hypersonic wake, one may adopt Eq. (52) for the spectral maximum, with an effective correlation length scale

$$\Lambda_\epsilon = \eta y_f \tag{54}$$

where y_f is the (statistically averaged) local half-width of the turbulent wake, and η is a proportional constant that may vary somewhat with location and with the aspect angle between the incident wave vector $\mathbf{k}_0$ and the wake axis x on account of anistropy. From the recent ballistic range study of Herrmann, Clay, and Slattery[27] on gas density fluctuations in the wake of hypersonic spheres using film contrast of schlieren photographs, one may deduce that the extent of anistropy was surprisingly slight and that the value of η seemed to vary between 0.05 and 0.1 over a great range of wake distance $10^3 < x/r_n < 10^5$. However, it should be noted that the film contrast correlation as described by Herrmann et al.[27] actually sampled the transverse curvature of the refractive index distribution along the averaged ray path within the turbulent wake volume, and hence tended to be heavily biased toward the smaller eddies as long as the density fluctuation associated with these eddies remained strong enough to be seen by the schlieren system. Thus, the observed homogeneity and isotropy in density fluctuation could well be restricted only to the smaller-scale structure of the wake turbulence, and the value of η so deduced could well represent a lower-bound estimate.

Again, for the purpose of making order-of-magnitude estimate, one may assume the fluctuations in number density Δn_ζ for all the molecular ions to be perfectly spatially correlated (justifiable from argument given in Sec. IV) with identical relative amplitude $\Delta n_\zeta/\bar{n}_\zeta$, such that

$$\left[\sum_\zeta \frac{m_e}{m_\zeta} Z_\zeta^2 \Delta n_\zeta\right]^2 \cong \left[\sum_\zeta \frac{m_e}{m_\zeta} Z_\zeta^2 \bar{n}_\zeta\right]^2 \overline{(\Delta n_\zeta)^2}/\bar{n}_\zeta^2 \tag{55}$$

with these assumptions, and with $d^3r_1 = \pi y_f^2 dx$, one obtains the following expression for the maximum back-scattering cross section per unit length of an ion-dominated far wake,

$$\left(\frac{d\sigma_{ii}}{dx}\right)_{\max} \cong 8\left(\frac{4\pi^2 Ry}{c}\right)^4 a_0^6 \left[\sum_\zeta \frac{m_e}{m_\zeta} Z_\zeta^2 \bar{n}_\zeta\right]^2 \times \left[\frac{\overline{(\Delta n_\zeta)^2}}{\bar{n}_\zeta^2}\right] \eta^3 g y_f^5 \tag{56}$$

By letting $[(\Delta n_\zeta)^2/\bar{n}_\zeta^2]\eta^3 g = 10^{-3}$, and using an empirical wake growth law deduced earlier from ballistic range experiments,[9]

$$y_f/r_n = 18Re_\infty^{-1/4} + 0.8[(x/r_n)^{1/3} - 20^{1/3}] \tag{57}$$

this maximum cross section per unit length has been evaluated for the far wake of the hypothetical $r_n = 1$ ft hypersonic sphere cited in example B of Figs. 1 and 2. The numerical result, given in square meter of cross section per meter length of wake, is tabulated in Table 2 at several selected values of x. It is seen that the absolute scattering cross section for such an ion-dominated wake is generally many orders of magnitude smaller than that of a typical electron-dominated far wake.[3] Because of its weak scattering property, experimental observation of "pure ion" fluctuations is expected to be difficult, and such scattering may well be obscured by residual electron effects due to imperfect turbulent mixing (and hence incomplete attachment) not being considered in the present paper.[28] The presence of residual electrons in a strongly attaching far wake would no doubt require more careful consideration of the electron-ion correlation effect discussed in Sec. IV, as well as the effect of residual electron-electron auto-correlation.

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